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Entrainment Theory for Compressible, Turbulent Boundary Layers on Adiabatic Walls

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Nomenclature

C_f = local skin-friction coefficient
H = δ^*/θ
H_{1k} = $\int_0^\delta \frac{u}{u_e}dy \Big/ \int_0^\delta \frac{u}{u_e}\left(1-\frac{u}{u_e}\right)dy$
H_k = $\int_0^\delta \left(1-\frac{u}{u_e}\right)dy \Big/ \int_0^\delta \frac{u}{u_e}\left(1-\frac{u}{u_e}\right)dy$
H_1 = Δ/θ
$\bar{H}$ = $\int_0^\delta \frac{\rho}{\rho_e}\left(1-\frac{u}{u_e}\right)dy/\theta$
j = 0, 1 for two-dimensional and axisymmetrical flow, respectively
M = Mach number
r = recovery factor
R = radius of the body of revolution
T = static temperature
T_r = recovery temperature
T^* = Eckert's[15] reference temperature
u,v = velocity components
x,y = coordinates measured along and normal to the body surface
δ = boundary-layer thickness
δ^* = displacement thickness = $\int_0^\delta\left(1-\frac{\rho u}{\rho_e u_e}\right)dy$
Δ = $\delta-\delta^*$
μ^* = viscosity evaluated at T^*
ρ = density
θ = momentum thickness = $\int_0^\delta \frac{\rho u}{\rho_e u_e}\left(1-\frac{u}{u_e}\right)dy$

Subscript

e = conditions at the outer edge of the boundary layer

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Introduction

THERE are many situations where engineering designers are concerned only with the gross characteristics of boundary-layer flows and not the point-by-point variation of the flow variables. In such cases integral methods are often adopted due to their ease of implementation, speed of computation and relative accuracy. A particularly simple and accurate integral method and also one that is ideally suited for extension to more complicated situations is the entrainment theory of Head.[1] Although originally developed for two-dimensional, turbulent, incompressible boundary layers the entrainment theory has been successfully extended to include effects of three-dimensionality,[2] compressibility,[3–5] boundary-layer–inviscid flow interactions,[6] surface roughness,[7] and body rotation.[8,9] Standen[3] and So[4] based their extensions to compressible flow on different transformation theories that effectively reduce the compressible problem to an incompressible one whereas Green[5] developed a third transformation procedure as well as a direct method. Green found that the direct method compared much more favorably with his own data, taken downstream of a shock wave–boundary-layer interaction, than the transformation procedure. Based on available zero pressure gradient data, Green went on to present convincing arguments which cast serious doubts on the validity of existing transformations. Other authors[10,11] have also noted deficiencies in the existing transformations and Alber[10] indicated that certain transformations could lead to substantial errors at high Mach numbers. On the other hand, McDonald[11] suggested that an improved transformation might result from adopting a two-layer model for the compressible, turbulent boundary layer.

Based on the aforementioned information it appears that the most promising approach for developing an integral method for compressible, turbulent boundary layers is to consider a direct approach that is based on firm physical principles such as Green's[5] entrainment theory. The purpose of the present analysis is to establish a simplification of Green's direct compressible entrainment theory that is more in line with Head's[1] original theory for incompressible boundary layers. The results will be applicable to both two-dimensional and axisymmetrical flows.

Analysis

The continuity equation for steady, compressible, turbulent flow may be written in the form[12]

$$\partial(\rho u)/\partial x + \partial(\rho v)/\partial y + j[(\rho u/R)dR/dx] = 0$$

where the flow variables appear as time-averaged quantities. Integrating across the boundary layer gives

$$\frac{1}{\rho_e u_e}\frac{d}{dx}\left(\int_0^\delta \rho u dy\right) + \frac{j}{R}\frac{dR}{dx}\int_0^\delta \frac{\rho u}{\rho_e u_e}dy = \frac{1}{u_e}\left(u_e\frac{d\delta}{dx} - v_e\right) \quad (1)$$

where the right side is termed the dimensionless rate of entrainment, F. Expanding the left side of Eq. (1) and introducing the displacement thickness gives

$$\frac{d\Delta}{dx} = F + (M_e^2 - 1)\frac{\Delta}{u_e}\frac{du_e}{dx} - j\frac{\Delta}{R}\frac{dR}{dx} \quad (2)$$

which reduces to Green's[5] compressible entrainment equation under two-dimensional flow conditions. This relation plus the momentum integral equation

$$\frac{d\theta}{dx} = \frac{C_f}{2} - (H + 2 - M_e^2)\frac{\theta}{u_e}\frac{du_e}{dx} - j\frac{\theta}{R}\frac{dR}{dx} \qquad (3)$$

complete the system of integral equations.

Following Green,[5] it is assumed that the dimensionless rate of entrainment, under compressible flow conditions, has the same dependency as Head's assumption for incompressible flow. That is, for compressible flow,

$$F = 0.0306(H_{1k} - 3.0)^{-0.653} \qquad (4)$$

where H_{1k} is the "kinematic" form parameter assumed to be descriptive of the velocity profile in the outer portion of the compressible boundary layer. Some justification for this assumption is evident from the right side of Eq. (1) which is identical in form with the incompressible definition of F.

In order to extract a solution from Eqs. (2–4) it is necessary to relate the parameter H_{1k} to the compressible parameters appearing in these equations. Green[5] accomplished this by devising empirical relations for $H_{1k}(H_k)$ and $H_1(\bar{H})$ based on his own data. It was further shown by Green that Head's[1] correlation between form parameters did not provide an adequate representation for either set of data. However, the present authors have found that Head's relation provides a very satisfactory correlation for $H_{1k}(H_k)$ when compared with the representative data of Winter et al.[13] Additionally it was found that the following equality is representative of the Winter et al. data, $H_k = \bar{H}$. Based on these observations the authors recommend, for compressible flow, that Head's correlation between form parameters be adopted for the relation $H_{1k}(\bar{H})$. Since Standen[3] has provided an analytical approximation for Head's correlation, a convenient form for the $H_{1k}(\bar{H})$ relation is

$$H_{1k} = 1.535(\bar{H} - 0.7)^{-2.715} + 3.3 \qquad (5)$$

Equations (4) and (5) represent direct extensions of Head's entrainment relations to compressible flows.

Two additional relations between form parameters may be analytically determined by adopting a suitable temperature-velocity relationship. Based on a quadratic equation for temperature, Spence[14] showed that

$$H = (T_r/T_e)(\bar{H} + 1) - 1 \qquad (6)$$

for adiabatic wall conditions. With the additional assumptions of a power law velocity profile and a specific heat ratio of 1.4, Green[5] presented

$$\frac{H_1}{H_{1k}} = 1 + \frac{rM_e^2}{5}\left(\frac{H_1 - 1}{H_1 + 2}\right)\Big/\left[\frac{rM_e^2}{5} + \left(\frac{H_1 + 1}{2}\right)\right] \qquad (7)$$

However, it should be noted that Green did not employ Eq. (7) in his calculation procedure but instead relied on his empirical $H_{1k}(H_k)$ relation.

The system of equations is completed by adopting a suitable skin-friction law such as the Ludwieg-Tillmann relation modified for compressible flow,

$$C_f = 0.246\left(\frac{\rho_e u_e \theta}{\mu^*}\right)^{-0.268}\left(\frac{T_e}{T^*}\right)10^{-0.678\bar{H}} \qquad (8)$$

according to the reference temperature concept of Eckert.[15] The variation of viscosity with temperature was accounted for by Sutherland's equation.

Results and Conclusions

Figures 1–3 compare the present method and the method of Green[5] with experimental data obtained over a range of Mach numbers and pressure gradients. All data were obtained for adiabatic wall conditions and all calculations were performed with a recovery factor of 0.89. Figure 1 compares local values of skin-friction coefficient for flow over flat plates, Fig. 2 compares values of H and θ for the favorable pressure gradient conditions investigated by Pasiuk et al.,[19] and Fig. 3 compares values of θ for the axisymmetrical flow conditions of Winter et al.[13] The authors attribute the improvements in accuracy achieved with the present method to a more accurate prediction of the form parameter $\bar{H}$.

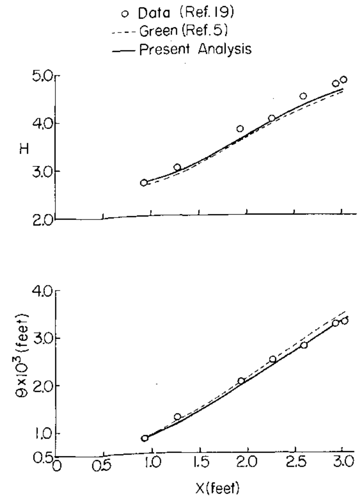

Fig. 2 Theoretical predictions of form parameter and momentum thickness compared with favorable pressure gradient data.

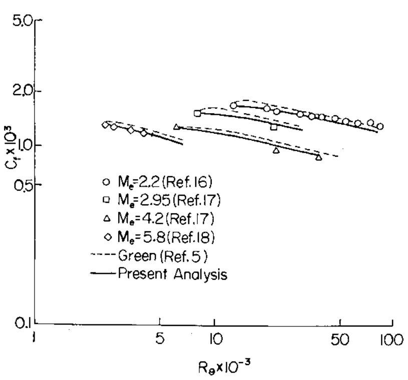

Fig. 1 Theoretical predictions of skin-friction coefficient compared with flat plate data obtained over a range of Mach numbers.

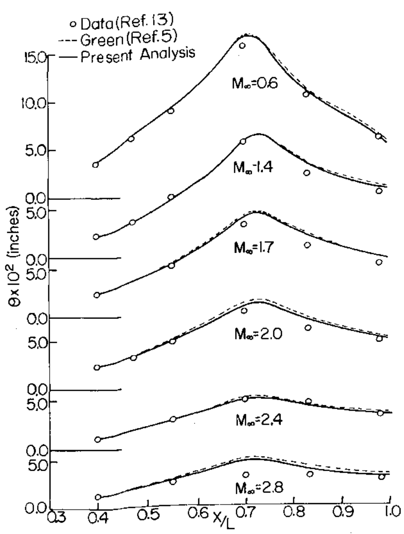

Fig. 3 Theoretical predictions of momentum thickness compared with data obtained on a body of revolution.

Based on the aforementioned results, it may be concluded that both the present method and the method of Green[5] provide adequate predictions of the development of turbulent, compressible boundary layers on adiabatic walls for the range of experimental conditions considered in Refs. 13 and 16–19. However, the present method has been shown to give an improvement in accuracy, relies less on empiricism and represents a simpler and more direct extension of Head's[1] incompressible entrainment theory to compressible flows.

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Impulse of a Ring with Nonlinear Material Behavior

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Introduction

THE linear dynamic response of nonaxisymmetric cylindrical shells has been considered to some detail in the literature. Payton[1] analyzed the impulsive membrane motion of a cylindrical shell and obtained a wave form of solution rather than the usual modal solution. The more general bending equations have been examined by Humphreys and Winter[2] using the modal technique.

The nonlinear dynamic analysis of shells and rings has almost exclusively concerned large displacements with linear stress-strain functions. Examples of analyses using nonlinear strain-displacement relationships may be found in Roth and Klosner[3] and Dowell.[4] In both of these studies, the nonlinear partial differential equations are reduced to nonlinear ordinary differential equations, using the Galerkin-Ritz technique. The analysis of perfectly plastic rate sensitive rings can be found in Perrone.[5] In his paper, Perrone obtains good correlation between approximate and numerical solutions.

The present Note considers the dynamic response of a ring whose stress-strain law is nonlinear and subject to a linear strain-displacement relationship. The results will, in general, be valid only for the first half cycle of motion. The initial conditions of the ring can be functions of the angular variable.

Equilibrium Equations and Problem Formulation

The geometric configuration is a thin walled ring of radius a, density ρ, and thickness h. For small displacements, the equilibrium equations describing membrane motion may be combined into a single equation relating hoop stress and hoop strain. This equation has been presented by Payton and is given by

$$(\partial^2\sigma/\partial\theta^2) - \sigma - \rho a^2(\partial^2\epsilon_\theta/\partial t^2) = 0 \qquad (1)$$

where σ is the hoop stress, ϵ_θ is the hoop strain, θ is the angular coordinate, and t is time. The ring is assumed to be fabricated from a work hardening material whose stress-strain relationship obeys the following nonlinear equation:

$$\sigma = E(\epsilon_\theta - b\epsilon_\theta^3) \qquad (2)$$

The constant E is analogous to Young's modulus, and b is a nondimensional positive constant. This relationship is valid until the absolute value of ϵ_θ is greater than $\bar{\epsilon}$; where $\bar{\epsilon}$ is the point of zero slope given by $\bar{\epsilon} = 1/(3b)^{1/2}$. The behavior of Eq. (2) is typical of a uniaxial tensile test during the loading phase. Whereas some substances (polymers for example) also obey this equation while unloading, most structural materials observe a linear stress-strain behavior during the unloading cycle. Thus, the total transient motion of the ring becomes a bookkeeping chore with different stress-strain behaviors. This study will be concerned primarily with the initial loading cycle and expressions developed will, in general, be valid only during this loading phase.

Eliminating σ from Eqs. (1) and (2), introducing the new variable ϵ, and defining the nondimensional time τ yields

$$(\partial^2\epsilon/\partial\theta^2) - \epsilon - (\partial^2\epsilon/\partial\tau^2) + \tfrac{1}{12}\{\epsilon^3 - [\partial^2(\epsilon^3)/\partial\theta^2]\} = 0 \qquad (3)$$

where $\tau = t(E/\rho)^{1/2}/a$ and $\epsilon = 2\epsilon_\theta/\bar{\epsilon}$. The variable ϵ appear-

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